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Non-Abelian nonlinear lattice equations on finite interval

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Abstract. We apply the inverse spectral problem method to the class of non-Abelian nonlinear lattice equations on the finite interval. The integrable discrete nonlinear Schrödinger and discrete modified Korteveg–de Vries equations are considered as examples. In the latter case the large time asymptotics for solutions are found.

1. Introduction

In recent years much attention has been paid to an investigation of boundary-value problems for integrable nonlinear equations, both partial differential and differential-difference [1–4].

The famous work of Moser [5], where the finite Toda lattice with free ends has been integrated, can be viewed as a first result in this direction for integrable lattice equations. Berezansky [6, 7] used the inverse spectral problem for Jacobi matrices [8] to integrate the half-infinite Toda lattice with one end at $-\infty$. We should also mention in this connection [9], where the role played by orthogonal polynomials on the line in the investigation of half-infinite lattices has been first recognized and [10], where Toda flows on self-adjoint matrices in $\ell_2(\mathbb{Z}_+)$ have been investigated.

The analogue of the direct and inverse spectral problem for self-adjoint difference expressions with operator coefficients (operator Jacobi matrices) allowed the application of the methods of [6, 7] to the investigation of isospectral deformations of operator Jacobi matrices [11, 12] and furthermore, after a proper generalization of the spectral theory to the non-symmetric case, to consider in [13, 14] Lax equations

$$\dot{L} = [L, A] = LA - AL \qquad \cdot = \frac{\mathrm{d}}{\mathrm{d}t} \tag{1.1}$$

where $L = L(t)$ is a finite or half-infinite difference expression with operator coefficients acting on sequences (u_n) of vectors from some Banach space

$$(Lu)_n = A_{n-1}u_{n-1} + B_n u_n + u_{n+1} \tag{1.2}$$

where $n = 0, \ldots, N \leqslant \infty$; $u_{-1} = 0$, $A_{-1} = 0$, A_n is invertible ($n \geqslant 0$) and $u_{N+1} = 0$, $A_N = 0$, if $N < \infty$.

An auxiliary operator A in (1.1) is a finite-difference expression with operator coefficients. Non-Abelian Toda and Volterra lattices (cf [15, 16]) are particular cases of (1.1). These equations were integrated in [13, 14] in the finite and half-infinite situation.

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The main point in the integration of the Lax equation (1.1) is to consider an evolution of the so-called moment sequence $S = (S_n)_{n=0}^{\infty}$, that corresponds to the difference expression (1.2):

$$S_n = (L^n)_{00}\,.$$

It turns out that S_n satisfy linear equations

$$\dot{S}_n = \sum_{j=0}^{k_1} M_j S_{n+j} + \sum_{j=0}^{k_2} S_{n+j} N_j \qquad n = 0, \ldots \tag{1.3}$$

where numbers $k_1,\ k_2$ and operator coefficients $M_j,\ N_j$ are determined by the auxiliary operator A. Generally speaking, $M_j,\ N_j$ depend on unknown functions $A_n,\ B_n$. However, in many situations it is possible to solve system (1.3) and then to restore unknown functions $A_n,\ B_n$, using the inverse spectral problem.

It is interesting to impose certain additional restrictions on coefficients $A_n,\ B_n$ in (1.2) and to consider the Lax equations which are compatible with these restrictions. In such a way one obtains reductions of non-Abelian integrable lattices. One of the possible reductions of the Lax equation (1.1) in the finite case is considered in section 2. (The half-infinite situation has been treated in [17].) In section 3 we study the spectral problem for the corresponding difference expression (1.2) and in section 4 give a recipe for solving the initial boundary-value problem for the non-Abelian equations introduced in section 2 with particular boundary conditions compatible with the Lax equation. These results are applied in section 5 to the investigation of the initial boundary-value problem on the finite interval for the discrete modified KdV (DMKdV) equation.

Section 6 is devoted to the investigation of the integrable version of the discrete nonlinear Schrödinger (DNLS) equation

$$\mathrm{i}\dot{r}_n = (1 - |r_n|^2)(r_{n-1} + r_{n+1}) - 2r_n \tag{1.4}$$

which is also known as the Ablowitz–Ladik model, since Ablowitz and Ladik introduced this equation and studied it by means of the inverse scattering problem method in the double-infinite case [18]. We should emphasize the difference between (1.4) and the standard discretization of the nonlinear Schrödinger equation, which appears in physics in the form

$$\mathrm{i}\dot{r}_n = (r_{n-1} + r_{n+1} - 2r_n) \pm |r_n|^2 r_n \tag{1.5}$$

and is non-integrable. The comparison of different features of the integrable and non-integrable DNLS equations can be found, e.g., in [19].

Finally section 7 treats other boundary conditions similar to those considered in [1, 20] for the half-infinite Toda lattice.

We would like to point out the close connection between our approach and methods of papers of Common and Hafez [21] and Common [22], where properties of various classes of continued fractions and related moment problems were used to linearize half-infinite DMKdV, DNLS and some other lattice equations.

2. Lax equation

We consider the Lax equation

$$\dot{L} = [L,\ A] \tag{2.1}$$

associated with the following difference expression with matrix coefficients

$$(Lu)_n = (\mathbf{1} - R_{n-1}^2)u_{n-1} + (R_{n-1} - R_n)u_n + u_{n+1}$$

$$u_{-1} = u_{N+1} = 0 \qquad u = (u_n)_{n=0}^N \quad u_n \in \mathbb{C}^m \qquad R_n = R_n(t) \in \mathbb{C}^{m\times m}. \tag{2.2}$$

We also assume that the following additional conditions are valid:

$$\mathbf{1} - R_n^2 \text{ is invertible } n = 0, \ldots, N-1 \tag{2.3}$$

and

$$R_{-1}^2 = R_N^2 = \mathbf{1}. \tag{2.4}$$

In what follows we identify the difference expression L with block 3-diagonal matrix acting in $\mathbb{C}^{(N+1)m}$. The auxiliary operator A is chosen in the form

$$(Au)_n = C_n u_n + D_n u_{n+1} + E u_{n+2} \qquad C_n = C_n(t) \qquad D_n = D_n(t) \qquad n = 0, \ldots, N. \tag{2.5}$$

Here E does not depend on t.

It follows from (2.1) and (2.3) that matrices $C_0,\ D_0,\ E$ and coefficients of the difference expression (2.2) completely determine C_n $(n = 1, \ldots, N)$ and D_n $(n = 1, \ldots, N-1)$. Therefore the consistency of (2.5) with (2.1)–(2.4) depends upon an appropriate choice of $C_0,\ D_0,\ E$. If, in addition, we want the Lax equation to be local with respect to R_n, then it can be verified directly that $C_0,\ D_0,\ E$ should be chosen to satisfy the following conditions:

$$R_n E = \pm E R_n \quad n = 1, 0, \ldots, N \qquad D_0 = R_{-1}E - ER_1 \qquad C_0 = C - \{R_{-1},\ ER_0\} \tag{2.6}$$

where $C \in \mathbb{C}^{m\times m}$ does not depend on t and $[C, E] = 0$. The Lax equation implies now that

$$D_n = R_{n-1}E - ER_{n+1} \qquad C_n = C - \{R_{n-1},\ ER_n\}$$

and is equivalent to the nonlinear system of differential equations

$$\begin{aligned} &\dot{R}_n = (\mathbf{1} - R_n^2)R_{n-1}E - ER_{n+1}(\mathbf{1} - R_n^2) + [R_n, C] \qquad n = 0, \ldots, N-1 \\ &\dot{R}_{-1} = [R_{-1}, C] \qquad \dot{R}_n = [R_N, C] \qquad R_{-1}^2 = R_N^2 = \mathbf{1}. \end{aligned} \tag{2.7}$$

Obviously, the change of variables $R_n \longrightarrow UR_nU^{-1}$, where $\dot{U} = CU$, leads to the system

$$\begin{aligned} &\dot{R}_n = (\mathbf{1} - R_n^2)R_{n-1}E - ER_{n+1}(\mathbf{1} - R_n^2) \qquad n = 0, \ldots, N-1 \\ &R_{-1} \equiv \Phi_1 \quad R_N \equiv \Phi_2 \qquad \Phi_1, \Phi_2 \in \mathbb{C}^{m\times m} \quad \Phi_1^2 = \Phi_2^2 = \mathbf{1} \end{aligned} \tag{2.8}$$

and therefore without loss of generality we may assume $C = 0$.

3. Spectral data and inverse problem

In order to apply the inverse spectral problem method to the system (2.8) let us consider spectral data that correspond to the difference expression (2.2) viewed as a linear operator in $\mathbb{C}^{(N+1)m}$.

For any $\lambda \in \mathbb{C}$ one can construct the set of matrix polynomials $P_0(\lambda) = \mathbf{1}, \ldots, P_n(\lambda)$ by means of the recursion

$$\lambda P_n(\lambda) = (\mathbf{1} - R_{n-1}^2)P_{n-1}(\lambda) + (R_{n-1} - R_n)P_n(\lambda) + P_{n+1}(\lambda) \qquad n = 0, \ldots, N-1. \tag{3.1}$$

Then $P_n(\lambda)$ is a polynomial of degree n

$$P_n(\lambda) = \lambda^n \mathbf{1} + (R_{n-1} - R_{-1})\lambda^{n-1} + \cdots.$$

Let

$$P_{N+1}(\lambda) = (\lambda \mathbf{1} + R_N - R_{N-1})P_N(\lambda) - (\mathbf{1} - R_{N-1}^2)P_{N-1}(\lambda) = \lambda^{N+1}\mathbf{1} + \sum_{j=0}^{N} F_j \lambda^j. \tag{3.2}$$

It is easy to see that the point λ belongs to the spectrum of L if and only if

$$\det\ P_{N+1}(\lambda) = 0. \tag{3.3}$$

Due to the isospectral property of the Lax equation, the scalar polynomial $\det P_{N+1}(\lambda)$ conserves, if R_n ($n = 0, \ldots, N-1$) evolve in accordance with (2.7) or (2.8). In fact, we shall show later that the polynomial $P_{N+1}(\lambda)$ itself is an integral of motion of the system (2.8).

Following [13, 14] we consider the sequence $S = (S_n)_{n=0}^{\infty}$ of matrices from $\mathbb{C}^{m \times m}$

$$S_n = (L^n)_{00} \quad n = 1, 2, \ldots \qquad S_0 = \mathbf{1} \tag{3.4}$$

where $(L^n)_{00}$ is a coefficient of u_0 in the expression for $(L^n u)_0$ or, in other words, the upper left block $m \times m$ coefficient of the matrix L^n. We shall call S a moment sequence of L. To justify the term 'moment', let us consider (2.2) under the additional assumptions

$$R_n = R_n^* \quad n = 1, \ldots, N \qquad R_N^2 < \mathbf{1} \quad n = 0, \ldots, N-1. \tag{3.5}$$

We assume also that

$$[R_n^2,\ R_k] = 0 \qquad n, k = -1, 0, \ldots, N. \tag{3.6}$$

Then L is similar to the self-adjoint operator $\tilde{L} = V^{-1}LV$, where V is a block diagonal operator in $\mathbb{C}^{(N+1)m}$:

$$(Vu)_n = V_n u_n \qquad V_0 = \mathbf{1} \qquad V_n = (\mathbf{1} - R_0^2)^{1/2} \cdots (\mathbf{1} - R_{n-1}^2)^{1/2} \quad n = 1, \ldots, N$$

and $\tilde{L}$ has a form

$$(\tilde{L}u)_n = (\mathbf{1} - R_{n-1}^2)^{1/2} u_{n-1} + (R_{n-1} - R_n)u_n + (\mathbf{1} - R_n^2)^{1/2} u_n.$$

According to the theory of symmetric second-order difference expressions with matrix coefficients [8, 23] there exists a set of non-negative $m \times m$ matrices $\rho_1, \cdots, \rho_K$ such that

$$S_n = \sum_{j=1}^{K} \lambda_j^n \rho_j \qquad n = 0, 1, \ldots \tag{3.7}$$

where $\lambda_1, \ldots, \lambda_K$ are distinct points of the spectrum of $\tilde{L}$ (and therefore of L). The representation (3.6) means that S_n is an nth moment of the non-negative matrix measure with points of growth λ_j and jumps $\rho_j = \rho_j(\lambda_j)$.

Given a sequence $S = (S_n)_{n=0}^{\infty}$ of $m \times m$ matrices, a natural question arises: when S is a moment sequence of the difference expression (2.2) which satisfies (2.3), (2.4).

Theorem 1. The sequence $S = (S_n)_{n=0}^{\infty}$ of $m \times m$ matrices is the moment sequence of some difference expression (2.2)–(2.4) with R_{-1}, R_N given if and only if the following conditions hold true.

(1) S is non-degenerate, i.e. for every $n = 0, 1, \dots, N$ the block Hankel matrix

$$H_n = (S_{j+k})_{j,k=0}^{n} \tag{3.8}$$

is invertible;

(2)

$$S_{2n+2} = \{R_{-1},\ S_{2n+1}\} \qquad n = 0, 1, \dots \tag{3.9}$$

(3) for every $k = 0, 1, \dots$

$$S_{N+k+1} = -\sum_{j=0}^{N} F_j S_{k+j} \tag{3.10}$$

where $m \times m$ matrices $F_0, \dots, F_N$ satisfy the relations

$$\begin{aligned} &R_N F_{N-2k+1} - F_{N-2k+1} R_{-1} = F_{N-2k} \\ &R_N F_{N-2k} + F_{N-2k} R_{-1} = 0 \qquad k = 0, \dots, \left[\frac{N}{2}\right] \qquad F_{N+1} = \mathbf{1},\ F_{-1} = 0 . \end{aligned} \tag{3.11}$$

The sketch of the proof will be given in the appendix.

Remarks.

(1) Conditions (3.8) and (3.10) ensure S to be the moment sequence of some difference expression (1.2) (see [13, 14]), whereas relations (3.9) and (3.10) are responsible for this difference expression to be of the form (2.2)–(2.4).

(2) Matrices $F_0, \dots, F_n$ in (3.10) coincide with coefficients of the polynomial $P_{N+1}(\lambda)$ (equation (3.2)), which correspond to L.

(3) Assumptions (3.5) and (3.6) imply the positive-definiteness of H_N in (3.8).

The inverse problem for (2.2) consists in recovering coefficients R_n from the given moment sequence which meets conditions of theorem 1. The recurrent procedure for the solution of the inverse problem in the case of general difference expression (1.2) was presented in [13, 14]. Another approach, proposed recently in [24], is based on the inversion of block Hankel matrices (3.8). Taking an advantage of the special form of (2.2), one can show that if

$$\Gamma_n = (\gamma_{jk}^{(n)})_{j,k=0}^{n} \qquad \gamma_{jk}^{(n)} \in \mathbb{C}^{m\times m}$$

is an inverse of H_n, then the formulae

$$R_{n-1} = R_{-1} + (\gamma_{nn}^{(n)})^{-1} \gamma_{n,n-1}^{(n)} \qquad n = 0, \dots, N \tag{3.12}$$

give the coefficients of the corresponding difference expression (2.2). As we shall show below, in particular cases (3.12) may be rewritten in a more explicit way.

4. Evolution of spectral data

Let us assume now that R_n $(n = 0, \ldots, N-1)$ evolve in accordance with nonlinear system (2.8) or, equivalently, the difference expression (2.2) satisfies the Lax equation (2.1). Consider the moment sequence $S = (S_n)_{n=0}^{\infty}$ corresponding to L. Then

$$\dot{S}_n = (\dot{L^n})_{00} = ([L^n, A])_{00} = [S_n, C_0] - D_0(L^n)_{10} - E(L^n)_{20}\,. \tag{4.1}$$

Using evident equalities

$$S_{n+1} = (L\cdot L^n)_{00} = (R_{-1} - R_0)S_n + (L^n)_{10} \qquad S_{n+2} = (L^2)_{00}S_n + (L^2)_{01}(L^n)_{10} + (L^2)_{02}(L^n)_{20}$$

and bearing in mind (2.2) and (2.6), one gets after excluding $(L^n)_{10}$, $(L^n)_{20}$ from (4.1)

$$\dot{S}_n = -S_n\{R_{-1}, ER_0\} + R_{-1}\{E, R_{-1}\}S_n + [E, R_{-1}]S_{n+1} - ES_{n+2}\,. \tag{4.2}$$

Direct calculations show that (4.2) is consistent with (3.9). It follows from (2.6) that (4.2) reads

$$\dot{S}_n = -S_n E\{R_{-1}, R_0\} + 2ES_n - ES_{n+2} \tag{4.3}$$

if E and R_n commute, and

$$\dot{S}_n = -S_n E[R_0, R_{-1}] + 2ER_{-1}S_{n+1} - ES_{n+2} \tag{4.4}$$

if E and R_n anticommute.

Lemma 1. Let coefficients of L satisfy the nonlinear system (2.8). Then

$$\dot{P}_{N+1}(\lambda) = 0$$

where polynomial $P_{N+1}(\lambda)$ is defined by (3.2).

Proof. We have to prove that coefficients F_j, $j = 0, \ldots, N$ of the polynomial $P_{N+1}(\lambda)$ are integrals of motion of the system (2.8). If L corresponds to the solution of (2.8), then by theorem 1, the block Hankel matrix $H_N = (S_{j+k})_{j,k=0}^{N}$ is invertible; therefore, it is enough to show that

$$\sum_{j=0}^{N} \dot{F}_j S_{k+j} = 0 \qquad k = 0, \ldots, N\,.$$

Differentiating and taking into account (4.2) we have

$$\sum_{j=0}^{N} \dot{F}_j S_{k+j} = \sum_{j=0}^{N}([F_j, R_{-1}\{E, R_{-1}\}]S_{k+j} + [F_j, [E, R_{-1}]]S_{k+j+1} - [F_j, E]S_{k+j+2})\,. \tag{4.5}$$

If $[R_n, E] = 0$ $(n = 1, 0, \ldots, N)$, then it is easy to see that $[F_n, E] = 0$ $(n = 0, \ldots, N)$ and the right-hand side of (4.5) is equal to zero. In the other case, $R_nE = -ER_n$ and the right-hand side of (4.5) can be rewritten as follows:

$$\begin{aligned}\sum_{j=0}^{N}(2[F_j, ER_{-1}]S_{k+j+1} - [F_j, E]S_{k+j+2}) &= 2[F_0, ER_{-1}]S_{k+1} \\ &\quad - [F_N, E]S_{k+N+2} + \sum_{j=1}^{N}(2[F_j, ER_{-1}] - [F_{j-1}, E])S_{k+j+1} \\ &\overset{(3.10)}{=} \sum_{j=0}^{N}(2[F_j, ER_{-1}] - [F_{j-1}, E] + [F_N, E]F_j)S_{k+j+1}\,.\end{aligned}$$

It follows from (3.1) and (3.2) that

$$F_{N-m}E = (-1)^{m+1}EF_{N-m}\,.$$

Therefore, by (3.11),

$$2[F_{N-2m}, ER_{-1}] - [F_{N-2m-1}, E] + [F_N, E]F_{N-2m} = -2E(F_{N-2m}R_{-1} + R_N F_{N-2m}) = 0$$

$$2[F_{N-2m+1}, ER_{-1}] - [F_{N-2m}, E] + [F_N, E]F_{N-2m+1}$$
$$= -2E(F_{N-2m} + F_{N-2m+1}R_{-1} - R_N F_{N-2m+1}) = 0\,.$$

Thus lemma 1 is proved.

Lemma 2. Let coefficients of L satisfy the nonlinear system (2.8). Then corresponding moments $S_n = (L^n)_{00}$, $n = 0, 1, \ldots$ can be found by formulae

$$S_n = \tilde{S}_n \tilde{S}_0^{-1} \tag{4.6}$$

where matrix functions $\tilde{S}_0, \ldots, \tilde{S}_N$ form a solution of the finite linear differential system with constant coefficients

$$\dot{\tilde{S}}_n = 2E\tilde{S}_n - E\tilde{S}_{n+2} \qquad n = 0, \ldots, N \tag{4.7}$$

if $ER_n = R_nE$ $(n = 1, \ldots, N)$ or

$$\dot{\tilde{S}}_n = -2ER_{-1}\tilde{S}_{n+1} - E\tilde{S}_{n+2} \qquad n = 0, \ldots, N \tag{4.8}$$

if $ER_n = -R_nE$, with initial data $\tilde{S}_n(0) = S_n(0)$, and

$$\tilde{S}_{N+k+1} = -\sum_{j=0}^{N} F_j \tilde{S}_{k+j} \qquad k = 0, 1, \ldots\,. \tag{4.9}$$

Proof. Let us denote by X the solution of the equation $\dot{X} = \{R_{-1}, ER_0\}X$ with initial condition $X(0) = \mathbf{1}$. Then functions $\tilde{S}_n = S_nX$ satisfy (4.7) or (4.8), depending on the sign in the first equality of (2.6), for $n = 0, 1, \ldots$. The identity $S_n = \mathbf{1}$ implies that $\tilde{S}_0 = X$, hence (4.6) holds. Formula (4.9) follows from (3.10) and, in particular, it gives a representation of $\tilde{S}_{n+1}$, $\tilde{S}_{n+2}$ as linear functions of $\tilde{S}_0, \ldots, \tilde{S}_n$ with coefficients depending only on $F_0, \ldots, F_N$. Together with lemma 1, this allows us to consider (4.7) and (4.8) as a finite linear systems with constant coefficients.

Linear systems (4.7) and (4.8) give us a linearization of the nonlinear system (2.8). The recipe for a solution of the initial boundary problem with boundary conditions R_{-1}, R_N, $(R_{-1}^2 = R_N^2 = \mathbf{1})$ and initial data $R_n(0)$ $(n = 0, \ldots, N-1)$ is to construct the difference expression $L(0)$ and the moment sequence $S(0) = (S_n(0))_{n=0}^{\infty}$, to find $S_n(t)$ by (4.6)–(4.9) and then to restore unknown functions $R_n(t)$ using (3.12), provided non-degeneracy conditions (3.8) is fulfilled. Note, that since the initial sequence $S(0)$ is non-degenerate, (3.8) may fail for at most countably many isolated values of t.

5. Discrete modified KdV equation

In the simplest case

$$m = 1 \qquad R_n = r_n \in \mathbb{C}^1 \qquad E = -1$$

the Lax equation (2.1) or, equivalently the system (2.8), turns into the finite discrete modified KdV (DMKdV) equation

$$\begin{aligned} &\dot{r}_n = (1 - r_n^2)(r_{n+1} - r_{n-1}) \qquad n = 0, \ldots, N-1 \\ &r_{-1} \equiv \varepsilon_1 \qquad r_N = \varepsilon_2 \qquad \varepsilon_{1,2} \in \{-1, 1\}. \end{aligned} \tag{5.1}$$

(Another Lax representation of (5.1), connected with the non-Abelian Volterra lattice can be found in [13].)

Since coefficients of the difference expression (2.2) that corresponds to (5.1) as well as moments $s_n = S_n$ are scalar, (3.12) can be simplified:

$$r_n = r_{-1} - \frac{\Delta_n'}{\Delta_n} \qquad n = 0, \ldots, N \tag{5.2}$$

where $\Delta_n = \det\ H_n = \det(s_{j+k})_{j,k=0}^n$ and

$$\Delta_n' = \det \begin{bmatrix} s_0 & \cdots & s_{n-1} & s_{n+1} \\ s_1 & \cdots & s_n & s_{n+2} \\ \cdots & \cdots & \cdots & \cdots \\ s_n & \cdots & s_{2n-1} & s_{2n+1} \end{bmatrix}. \tag{5.3}$$

Note that due to (3.9) moments s_n satisfy relations

$$s_{2n+2} = 2r_{-1}s_{2n+1} \qquad n = 0, 1, \ldots. \tag{5.4}$$

Since L is a 3-diagonal matrix with scalar elements, and matrix elements above and below the main diagonal are non-zero due to (2.3), L has exactly $N+1$ distinct points of the spectrum: $\lambda_0, \ldots, \lambda_N$. However, these points could not be arbitrary, as coefficients of the polynomial $P_{N+1}(\lambda) = (\lambda - \lambda_0)\cdots(\lambda - \lambda_N) = \lambda^{N+1} + \sum_{j=0}^N F_j\lambda^j$ obey (3.11), which now reads

$$F_{N-2K} = (r_N - r_{-1})F_{N-2K+1} \qquad (r_N + r_{-1})F_{N-2K} = 0. \tag{5.5}$$

The relations (5.5) yield the following.

Lemma 3. If $r_N = r_{-1}$, then the spectrum of L is symmetric with respect to zero, zero being a point of spectrum if and only if N is even. If $r_N = -r_{-1}$, the spectrum of L consists of the point $2r_{-1}$ and the part which is symmetric with respect to zero, zero being a point of spectrum if and only if N is odd.

To find a solution of the initial boundary-value problem for DMKdV we use lemma 2 and (5.2). Taking into account the fact that r_n in (5.2) does not change after the multiplication of all moments s_n by the same number, we obtain the following.

Theorem 2. For any initial data $r_0(0), \ldots, r_{N-1}(0)$ $(1 - r_n^2(0) \neq 0)$ the solution of the initial boundary-value problem (5.1) for the DMKdV equation is given by (5.2) and (5.3) with s_n substituted by $z_n(t)$, where

$$z_{2n}(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\,(\exp(tL^2(0)))_{00} \qquad z_{2n+1}(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\,(L(0)\exp(tL^2(0)))_{00} \tag{5.6}$$

and $L(0)$ is a difference expression (2.2) constructed from initial data.

Proof. It follows from lemma 2, that for any t values $s_n(t)$ can be found as

$$s_n(t) = \tilde{s}_n(t)\tilde{s}_0(t)^{-1} = z_n(t)z_0(t)^{-1}$$

where

$$\dot{z}_n(t) = z_{n+2}(t), \; z_n(0) = s_n(0) \qquad n = 0, 1, \ldots .$$

It is easy to see that $z_n(t) = \sum_{k=0}^{\infty} S_{n+2k}(0)t^k/k!$. Bearing in mind (3.4), we get

$$z_0(t) = \sum_{k=0}^{\infty}(L^{2k}(0))_{00}\,\frac{tk}{k!} = \left(\sum_{k=0}^{\infty} L^{2k}(0)\,\frac{t^k}{k!}\right)_{00} = (\exp(tL^2(0)))_{00}$$

$$z_1(t) = (L(0)\exp(tL^2(0)))_{00}\,.$$

Then (5.6) follows immediately.

Generally speaking, the solution of (5.1) may have singularities. However, it is globally defined if initial data satisfy (3.5):

$$r_n(0) \in \mathbb{R}, \; r_n^2(0) < 1 \qquad n = 0, 1, \ldots, N-1\,. \tag{5.7}$$

Since (3.6) obviously holds, $L(0)$ is similar to a self-adjoint 3-diagonal matrix and therefore we have a representation (3.7) for the moments $s_n(0)$ $(n = 0, 1, \ldots)$

$$s_n(0) = \sum_{j=0}^{N} \lambda_j^n \rho_j \tag{5.8}$$

where $\lambda_0 < \lambda_1 < \cdots < \lambda_N \in \mathbb{R}$ are points of the spectrum of $L(0)$ and $\rho_j = \rho(\lambda_j) > 0$ are jumps of the scalar spectral measure $\rho(\lambda)$ with points of growth $\lambda_0, \ldots, \lambda_N$ and a property $\sum_{j=0}^{N} \rho(\lambda_j) = 1$. Using (5.4) and lemma 3, we get

$$(\lambda - 2r_{-1})\rho(\lambda) = -(\lambda + 2r_{-1})\rho(-\lambda) \tag{5.9}$$

for every $\lambda \in \mathbb{R}$. Since $\rho(\lambda_j) > 0$, (5.3) implies $|\lambda_j| \leqslant 2$ $(j = 0, \ldots, N)$.

It is easily seen from (5.8) that (5.6) can be rewritten as

$$z_n(t) = \sum_{j=0}^{N} \lambda_j^n \rho_j \, e^{\lambda_j^2 t}\,.$$

Moreover, it can be shown that determinants Δ_n, Δ_n' in (5.2) and (5.3) with S_n substituted by $z_n(t)$ have the following expressions in terms of λ_j, ρ_j:

$$\Delta_n = \Delta_n(t) = \sum_{0 \leqslant j_0 < \cdots < j_n \leqslant N} e^{(\lambda_{j_0}^2 + \cdots + \lambda_{j_n}^2)t} \rho_{j_0} \cdots \rho_{j_n} W^2(\lambda_{j_0}, \ldots, \lambda_{j_n}) \tag{5.10}$$

$$\Delta_n' = \Delta_n'(t) = \sum_{0 \leqslant j_0 < \cdots < j_n \leqslant N} e^{(\lambda_{j_0}^2 + \cdots + \lambda_{j_n}^2)t} \rho_{j_0} \cdots \rho_{j_n} W^2(\lambda_{j_0}, \ldots, \lambda_{j_n})\,(\lambda_{j_0} + \cdots + \lambda_{j_n})\,. \tag{5.11}$$

Theorem 3. Let initial data $r_n(0)$ $(n = 0, \dots, N-1)$ for (5.1) satisfy (5.7) and let $\lambda_0 < \lambda_1 < \cdots < \lambda_N$; ρ_j $(j = 0, \dots, N)$ are points of the spectrum and jumps of the spectral measure which correspond to the matrix $L(0)$, constructed from $r_n(0)$. Then

(1) Initial boundary-value problem (5.1) has a unique globally defined solution

$$r_n(t) = r_{-1} - \frac{\Delta_n'(t)}{\Delta_n(t)} \tag{5.12}$$

with $\Delta_n(t)$, $\Delta_n'(t)$ as in (5.10) and (5.11).

(2) Solution (5.12) has the following asymptotics as t tends to $+\infty$.

(i) If $r_N = r_{-1}$, then

$$r_{2n+1}(t) \longrightarrow r_N \qquad r_{2n}(t) \longrightarrow r_N - \frac{\mu_{N-n}^2}{2r_N} \qquad t \longrightarrow \infty \tag{5.13}$$

(ii) If $r_N = -r_{-1}$, then

$$r_{2n+1}(t) \longrightarrow r_N - \frac{\mu_{N-n}^2}{2r_N} \qquad r_{2n}(t) \longrightarrow r_N \qquad t \longrightarrow \infty. \tag{5.14}$$

Here

$$\mu_{N-n} = \begin{cases} \lambda_{N-n} & \text{if } \lambda_N < 2 \\ \lambda_{N-n-1} & \text{if } \lambda_N = 2. \end{cases}$$

Proof. The first statement of the theorem drops out immediately from theorem 2 and the fact that all terms in the sum (5.10) are positive.

To prove the second statement, one has to consider the leading term in (5.10), (5.11) as t tends to $+\infty$. By lemma 3, in the case (i) the set $\{\lambda_j;\ i = 0, \dots, N\}$ is symmetric with respect to zero, therefore the leading exponent in expressions for $\Delta_{2n}(t)$, $\Delta_{2n}'(t)$ is $e^{(2(\lambda_N^2+\cdots+\lambda_{N-n+1}^2)+\lambda_{N-n}^2)t}$, the leading exponent in expressions for $\Delta_{2n+1}(t)$, $\Delta_{2n+1}'(t)$ is $e^{2(\lambda_N^2+\cdots+\lambda_{N-n}^2)t}$. To prove (5.13), we need only to compute coefficients near the leading exponents, using (5.9). The case (ii) can be treated in the same way. Note that by lemma 3, $\lambda_N = 2$ if and only if $r_N = -r_{-1} = -1$.

Remark. Formulae analogous to (5.10) and (5.11) in the case of the finite non-periodic Toda lattice were first presented in [25] and used therein to rederive the Moser's result on the asymptotic behaviour of the finite Toda lattice [5]. In [26] they were used to find a t-series expansion of solutions of the double-infinite Toda lattice.

6. Discrete nonlinear Schrödinger equation

Another particular case of the Lax equation (2.1)–(2.5) is the finite DNLS equation

$$i\dot{r}_n = (1 - |r_n|^2)\,(r_{n-1} + r_{n+1}) - 2r_n \qquad n = 0, \dots, N-1 \tag{6.1}$$

with boundary conditions

$$r_{-1} = e^{2it+i\varphi_1} \qquad r_N = e^{2it+i\varphi_2} \qquad \varphi_1, \varphi_2 \in \mathbb{R}. \tag{6.2}$$

We obtain DNLS from (2.7) after choosing

$$m = 2 \qquad R_n = \begin{pmatrix} 0 & \bar{r}_n \\ r_n & 0 \end{pmatrix} \qquad E = -C = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}. \tag{6.3}$$

After a substitution $r_n \longrightarrow \mathrm{e}^{-2\mathrm{i}t} r_n$ (which is equivalent to putting C in (6.3) to be equal to zero) we get a system

$$\begin{aligned} &\mathrm{i}\dot{r}_n = (1 - |r_n|^2)\,(r_{n-1} + r_{n+1}) \qquad n = 0, \ldots, N-1 \\ &r_{-1} = \mathrm{e}^{\mathrm{i}\varphi_1} \qquad r_N = \mathrm{e}^{\mathrm{i}\varphi_2} \qquad \varphi_1, \varphi_2 \in \mathbb{R}^1. \end{aligned} \tag{6.4}$$

Before applying to (6.4) results of section 4, let us discuss, as in the previous section, spectral properties of the corresponding difference expression (2.2).

The moment sequence (3.4) now has the form

$$S_{2n+1} = \begin{pmatrix} 0 & \bar{s}_{2n+1} \\ s_{2n+1} & 0 \end{pmatrix} \qquad S_{2n} = s_{2n}\mathbf{1} \tag{6.5}$$

where $s_{2n} \in \mathbb{R}$ ($n = 0, 1, \ldots$) and due to (6.3) the condition (3.9) of theorem 1 reads

$$s_{2n+2} = 2\,\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\varphi_1} s_{2n+1}). \tag{6.6}$$

Given a sequence $(S_n)_n^\infty$, coefficients r_n can be found from formulae which follow from (3.12):

$$r_n = r_{-1} - \frac{\Delta'_n}{\Delta_n} \tag{6.7}$$

where

$$\begin{aligned} \Delta_n &= \det \begin{bmatrix} s_0 & \bar{s}_1 & s_2 & \bar{s}_3 & \cdot \\ s_1 & s_2 & s_3 & \cdot & \\ s_2 & \bar{s}_3 & \cdot & & \\ s_3 & \cdot & & & \\ \cdot & & & & s_{2n} \end{bmatrix} \\ \Delta'_{2n-1} &= \det \begin{bmatrix} s_0 & \bar{s}_1 & \cdot & s_{2n-2} & s_{2n} \\ s_1 & \cdot & & s_{2n-1} & s_{2n+1} \\ \cdot & & & & \\ s_{2n-2} & \bar{s}_{2n-1} & & & \\ s_{2n-1} & s_{2n} & & s_{4n-3} & s_{4n-1} \end{bmatrix} \\ \Delta'_{2n} &= \det \begin{bmatrix} s_0 & s_1 & \cdot & s_{2n-1} & s_{2n+1} \\ \bar{s}_1 & \cdot & & s_{2n} & s_{2n+2} \\ \cdot & & & & \\ \bar{s}_{2n-1} & & & & \\ s_{2n} & & & s_{4n-1} & s_{4n+1} \end{bmatrix}. \end{aligned} \tag{6.8}$$

The spectrum of L is always symmetric with respect to zero and the multiplicity of eigenvalues is less than or equal to 2. Since for R_n from (6.3) condition (3.6) is obviously fulfilled, assumption (3.5) or, equivalently,

$$|r_n|^2 < 1 \qquad n = 0, 1, \ldots, N-1 \tag{6.9}$$

leads to the representation (3.7) for moments S_n with

$$\rho_j = \rho(\lambda_j) = \begin{pmatrix} \rho_+(\lambda_j) & \overline{\rho_-(\lambda_j)} \\ \rho_-(\lambda_j) & \rho_+(\lambda_j) \end{pmatrix} \geqslant 0 . \tag{6.10}$$

It follows from (6.5) that

$$\rho_\pm(-\lambda_j) = \pm\rho_\pm(\lambda_j) .$$

Moreover, (6.6) yields the identity

$$\lambda_j \rho_+(\lambda_j) = 2\,\mathrm{Re}(e^{-i\varphi}\rho_-(\lambda_j)) . \tag{6.11}$$

Using the non-negativity of $\rho(\lambda_j)$, we get from (6.10), (6.11)

$$\rho_+(\lambda_j)^2 - |\rho_-(\lambda_j)|^2 = \left(\frac{4}{\lambda_j^2} - 1\right)(\mathrm{Re}(e^{-i\varphi}\rho_-(\lambda_j)))^2 - (\mathrm{Im}(e^{-i\varphi}\rho_-(\lambda_j)))^2 \geqslant 0 \tag{6.12}$$

which means that for every λ_j from the spectrum of L

$$|\lambda_j| \leqslant 2 . \tag{6.13}$$

Let us now return to the initial boundary-value problem (6.4) with initial data $r_n(0)$ $(n = 0, \ldots, N-1)$ and let $L(0)$ be the corresponding difference expression constructed by (6.3), (2.2) with moment sequence $(S_n(0))_{n=0}^{\infty}$, spectrum λ_j $(j = 0, \ldots, k)$ and in the case when $r_n(0)$ satisfies (6.9), with non-negative jumps of the spectral measure $\rho(\lambda_j)$. Let us also denote

$$\mu(\lambda) = \lambda\sqrt{4-\lambda^2} \qquad \alpha(\lambda) = \mathrm{Re}(e^{-i\varphi}\rho_-(\lambda)) \qquad \beta(\lambda) = \mathrm{Im}(e^{-i\varphi}\rho_-(\lambda)) .$$

Theorem 4. Solution of the initial boundary value problem (6.4) is given by formulae (6.7), (6.8) with s_n substituted by the solution $\tilde{s}_n$ of the linear system

$$\dot{\tilde{s}}_{2n+1}(t) = 2i(\tilde{s}_{2n+1}(t) + e^{2i\varphi}\overline{\tilde{s}_{2n+1}(t)}) - i\tilde{s}_{2n+3}(t) \qquad \tilde{s}_{2n+1}(0) = s_{2n+1}(0) \tag{6.14}$$

$$\tilde{s}_0(t) = \exp\left(-2\int_0^t \mathrm{Im}(e^{i\varphi}\tilde{s}_1(u))\,du\right) \qquad \tilde{s}_{2n+2}(t) = 2\,\mathrm{Re}(e^{-i\varphi}\tilde{s}_{2n+1}(t)) . \tag{6.15}$$

If initial data $r_n(0)$ $(n = 0, \ldots, N-1)$ satisfy (6.9), then the solution is globally defined and

$$\tilde{s}_{2n+1}(t) = \sum_{j=0}^{k} \lambda_j^{2n+1}\tilde{\rho}_-(\lambda_j, t) \tag{6.16}$$

where

$$\tilde{\rho}_-(\lambda_j, t) = e^{i\varphi}(\alpha(\lambda_j)\left(\cosh(\mu(\lambda_k)t) + \frac{i}{\lambda_j}\sqrt{4-\lambda_j^2}\,\sinh(\mu(\lambda_j)t)\right)$$
$$+ \beta(\lambda_j)\left(i\cosh(\mu(\lambda_j)t) + \frac{\lambda_j}{\sqrt{4-\lambda_j^2}}\,\sinh(\mu(\lambda_j)t)\right) . \tag{6.17}$$

Proof. (6.4) is equivalent to the Lax equation (2.1)–(2.5) with $E = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}$ anticommuting with $R_n = \begin{pmatrix} 0 & \bar{r}_n \\ r_n & 0 \end{pmatrix}$ and $C = 0$. Therefore, by lemma 2 the moment sequence $(S_n(t))_{n=0}^{\infty}$ corresponding to $L(t)$ can be found from (4.6), (4.8). It follows from the proof of lemma 2 and that $\dot{\tilde{S}}_0 = \{R_{-1}, ER_0\}\, \tilde{S}_0 = -2\,\mathrm{Im}(\mathrm{e}^{-\mathrm{i}\varphi} r_0)\tilde{S}_0\ \tilde{S}_0(0) = 1$. Hence, $\tilde{S}_n$ has the form (6.5) and since r_n in (6.7) does not change after the multiplication of all s_n by the same real number, we may substitute $S_n(t)$ by $\tilde{S}_n(t)$, where

$$\tilde{S}_{2n}(t) = \begin{pmatrix} \tilde{s}_{2n}(t) & 0 \\ 0 & \tilde{s}_{2n}(t) \end{pmatrix} \qquad \tilde{S}_{2n+1}(t) = \begin{pmatrix} 0 & \bar{\tilde{s}}_{2n+1}(t) \\ \tilde{s}_{2n+1}(t) & 0 \end{pmatrix}.$$

The linear system (6.14), (6.15) is simply system (4.8) rewritten using identity (6.6).

Now let initial data satisfy (6.9). It can be verified directly that functions $\tilde{s}_{2n+1}(t)$ defined by (6.14), (6.15) form a solution of (6.14). Moreover, (6.15) leads to the following representation for $\tilde{s}_{2n}(t)$:

$$\tilde{s}_{2n}(t) = \sum_{j=0}^{k} \lambda_j^{2n} \tilde{\rho}_+(\lambda_j, t)$$

where

$$\tilde{\rho}_+(\lambda_j, t) = \frac{2}{\lambda_j}\left(\alpha(\lambda_j)\cosh(\mu(\lambda_j)t) + \frac{\lambda_j}{\sqrt{4-\lambda_j^2}}\beta(\lambda_j)\sinh(\mu(\lambda_j)t)\right) = \frac{2}{\lambda_j}\,\mathrm{Re}(\mathrm{e}^{-\mathrm{i}\varphi}\tilde{\rho}_-(\lambda_j, t)).$$

Then

$$\tilde{s}_n(t) = \sum_{j=0}^{k} \lambda_j^n \tilde{\rho}(\lambda_j, t)$$

where

$$\tilde{\rho}(\lambda_j, t) = \begin{pmatrix} \tilde{\rho}_+(\lambda_j, t) & \bar{\tilde{\rho}}_-(\lambda_j, t) \\ \tilde{\rho}_-(\lambda_j, t) & \tilde{\rho}_+(\lambda_j, t) \end{pmatrix}.$$

In view of (6.12), (6.13), matrices $\tilde{\rho}(\lambda_j, t)$ are non-negative. This, together with the inequality

$$\sum_{j=0}^{k} \tilde{\rho}(\lambda_j, t) = \tilde{S}_0(t) = \mathbf{1} > 0$$

yields a positive-definiteness of the matrix H_N in (3.8). Therefore, by theorem 1, for every t we may restore $r_n(t)$ using (6.7), (6.8) which means that solution is globally defined.

7. Another boundary condition

In this section we suppose that assumption (3.6) is satisfied and consider system (2.8):

$$\dot{R}_n = (1 - R_n^2)(R_{n-1}E - ER_{n+1}) \qquad n = 1, \ldots, N \tag{7.1}$$

with boundary conditions

$$R_0 \equiv 0 \qquad R_{N+1} \equiv \Phi \qquad \Phi^2 = \mathbf{1}. \tag{7.2}$$

Let us denote

$$R_{-n} = -\varepsilon^n R_n \qquad n = 1, \ldots, N \tag{7.3}$$

where $\varepsilon = -1$, if $R_n E = E R_n$ and $\varepsilon = 1$ otherwise. Then it is not hard to see that functions $R_1, \ldots, R_N$ form a solution of the initial boundary-value problem (7.1), (7.2) if and only if functions $R_{-N}, \ldots, R_{-1},\ R_0 \equiv 0,\ R_1, \ldots, R_N$ form a solution of the initial boundary-value problem

$$\begin{aligned} &\dot{R}_n = (1 - R_n^2)(R_{n-1} E - E R_{n+1}) \qquad n = -N, \ldots, N \\ &R_{-N-1} \equiv -\varepsilon^{N+1} \Phi \qquad R_{N+1} \equiv \Phi . \end{aligned} \tag{7.4}$$

Therefore, in order to solve (7.1), (7.2) with initial data $R_1(0), \ldots, R_N(0)$ one can extend the initial data according to (7.3) and then use results of section 4 to solve (7.4).

Both finite DMKdV and DNLS equations with left-end boundary condition $r_0 = 0$, and right-end boundary condition being unchanged, can be treated in this way.

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Appendix

As we have mentioned, conditions (3.8) and (3.10) of theorem 1 are necessary and sufficient for the sequence $S = (S_n)_{n=0}^{\infty}$ to be a moment sequence of some finite difference expression (1.2) [13, 14]. We shall prove here that if this difference expression has a form (2.2)–(2.4), then corresponding moment sequence satisfies (3.9) and relations (3.11) for coefficients F_j in (3.10) hold true. The inverse implication can be checked straightforwardly.

Consider matrix polynomials $P_n(\lambda)$ ($n = 0, \ldots, N+1$) defined by the recursion (3.1)–(3.2). Then, by the usual induction, one gets the following.

Lemma 5. Let

$$P_n(\lambda) = \lambda^n \mathbf{1} + P_{n,n-1}\lambda^{n-1} + \cdots + P_{n,0} .$$

Then

$$\begin{aligned} &P_{n+1,n-2k} = R_n P_{n,n-2k} - P_{n,n-2k} R_{-1} \\ &P_{n+1,n-2k-1} = P_{n,n-2k-2} + R_n P_{n,n-2k-1} + P_{n,n-2k-1} R_{-1} . \end{aligned} \tag{A.1}$$

Since $F_j = P_{N+1,j}$ ($j = 0, \ldots, N$) relations (3.11) easily drops out from (A.1) and (2.4).

It follows from (3.1) and (3.4) that $\sum_{j=0}^{n} P_{n,j} S_j = 0$. Therefore

$$S_n = -\sum_{j=0}^{n-1} P_{n,j} S_j . \tag{A.2}$$

Suppose $S_{2k+2} = \{S_{2k+1}, R_{-1}\}$ $(k = 0, \cdots, n-1)$. Then $[R_{-1}, S_{2k+2}] = [R_{-1}, \{S_{2k+1}, R_{-1}\}] = [R_{-1}^2, S_{2k+1}] = 0$.

Consider

$$\begin{aligned} S_{2n+2} &\overset{(A.2)}{=} -\sum_{k=0}^{n}(P_{2n+2,2n-2k+1}S_{2n+1-2k} + P_{2n+2,2n-2k}S_{2n-2k}) \\ &\overset{(A.1)}{=} -R_{2n+1}\sum_{k=0}^{2n+1}P_{2n+1,k}S_k + R_{-1}S_{2n+1} \\ &\quad + \sum_{k=0}^{n-1}P_{2n+1,2n-2k-1}(R_{-1}S_{2n-2k-1} - S_{2n-2k}) - \sum_{k=0}^{n}P_{2n+1,2n}R_{-1}S_{2n-2k}\,. \end{aligned}$$

The first sum in the last expression is equal to zero according to (A.2). Using the assumption of the induction and commutativity of R_{-1} and S_{2n-2k} $(k = 0, \ldots, n)$, we obtain

$$S_{2n+2} = R_{-1}S_{2n+1} - \sum_{k=0}^{2n}P_{2n+1,k}S_kR_{-1} \overset{(A.2)}{=} \{R_{-1}, S_{2n+1}\}\,.$$

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